Basic Symplectic Topology

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Contents

I. Introduction

The goal of these notes is to fill-in details and provide solutions to selected exercises in McDuff and Salamon's book, *Introduction to Symplectic Topology*. I may also draw from other sources, when relevant, as there are some arguments in McDuff-Salamon which rely on material that warrants further discussion for a novice (such as myself).

II. Chapter 1

A. Notes

I would like to begin these notes to filling in a few details which are glossed-over in the discussion of the Lagrangian-Hamiltonian correspondence of McDuff-Salamon. It is straightforward to demonstrate that a path $x(t)$ is a critical point of the action functional corresponding to Lagrangian L if and only if it is a solution to the Euler-Lagrange equation,

$$\frac{d}{dt}\frac{dL}{dv_i}(t, x(t), \dot{x}(t)) - \frac{dL}{dx_i}(t, x(t), \dot{x}(t)) = 0 \tag{1}$$

over all $i = 1, \dots, n$. Note that

$$\frac{d}{dt}\frac{dL}{dv_j}(t, x(t), \dot{x}(t)) = \frac{d}{ds}\frac{dL}{dv_j}(s, x(t), \dot{x}(t))\bigg|_{s=t} + \sum_i \left[\dot{x}(t)\frac{d^2 L}{dx_i dv_j}(t, x(t), \dot{x}(t)) + \ddot{x}(t)\frac{d^2 L}{dv_i dv_j}(t, x(t), \dot{x}(t))\right] \tag{2}$$

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In the case that we have the Legendre condition on L, as explained in the book, then the above is a system of second-order differential equations for $x(t)$. As per usual, we may convert a system of second-order differential equations into a larger first-order system. Usually, we do this by setting $y = \dot{x}$, however, in this case, we can be somewhat more tactful and utilize the Legendre transform, where we define the functions $y_k(t, x, v) = \frac{dL}{dv_k}(t, x, v)$, and we use the notation $y_k(t) = y_k(t, x(t), \dot{x}(t)) = \frac{dL}{dv_k}(t, x(t), \dot{x}(t))$ for some trajectory $x(t)$. It then follows that $x(t)$ being a solution to the original Euler-Lagrange equation is equivalent to the condition that

$$\dot{y_k}(t) = \frac{d}{dt}\frac{dL}{dv_k}(t, x(t), \dot{x}(t)) = \frac{dL}{dx_k}(t, x(t), \dot{x}(t)) \tag{3}$$

Suppose that the Legendre condition holds at some point (t_0, x_0, v_0) with $y_0 = \frac{dL}{dv}(t_0, x_0, v_0)$, then it follows from implicit function theorem that in a neighbourhood of this point there are unique smooth functions $G_k(t, x, y)$ such that $y - \frac{dL}{dv}(t, x, G(t, x, y)) = 0$ in a neighbourhood U of (t_0, x_0, y_0). We can use the notation $v_k = G_k(t, x, y)$ as shorthand. It follows that on U, we can define a *Hamiltonian* $H : U \to \mathbb{R}$ as

$$H(t, x, y) = \sum_{j=1}^{n} y_j G_j(t, x, y) - L(t, x, G(t, x, y)) = \sum_{k=1}^{n} y_j v_j - L \tag{4}$$

Note that

$$\frac{dH}{dy_j}(t, x, y) = G_j(t, x, y) + \sum_{k=1}^{n} \frac{dG_k}{dy_j}\left[y_k - \frac{dL}{dv_k}(t, x, G(t, x, y))\right] = G_j(t, x, y) \tag{5}$$

and

$$\frac{dH}{dx_j}(t, x, y) = \sum_{k=1}^{n} \frac{dG_k}{dx_j}\left[y_k - \frac{dL}{dv_k}(t, x, G(t, x, y))\right] - \frac{dL}{dx_j} = -\frac{dL}{dx_j}(t, x, G(t, x, y)) \tag{6}$$

on U. It follows immediately that if $x(t)$ is a solution to the Euler-Lagrange equation, and the Legendre condition is satisfied at $(t_0, x(t_0), \dot{x}(t_0))$, then in a neighbourhood U of $(t_0, x(t_0), y(t_0))$, the Hamiltonian will be defined. We can choose t sufficiently close to t_0 (i.e. in $(t_0 - \varepsilon, t_0 + \varepsilon)$) so that $(t, x(t), y(t)) \in U$, and we will have

$$\frac{dH}{dy_j}(t, x(t), y(t)) = G_j(t, x(t), y(t)) \quad \text{and} \quad \frac{dH}{dx_j}(t, x(t), y(t)) = -\frac{dL}{dx_j}(t, x(t), G(t, x(t), y(t))) \tag{7}$$

By definition, $y(t) = \frac{dL}{dv}(t, x(t), \dot{x}(t))$, so both $t \mapsto G(t, x(t), \dot{x}(t))$ and $t \mapsto \dot{x}(t)$ are smooth functions on $(t_0 - \varepsilon, t_0 + \varepsilon)$ such that $y(t) - \frac{dL}{dv}(t, x(t), v(t)) = 0$ when plugged-in as $v(t)$. However, the Legendre condition and implicit function theorem imply that such a $v(t)$ in a neighbourhood of t_0 is unique, so on interval $(t_0 - \delta, t_0 + \delta)$, we have $\dot{x}(t) = G(t, x(t), y(t))$, and using Eq. (3), we have

$$\frac{dH}{dy_j}(t, x(t), y(t)) = \dot{x}_j(t) \quad \text{and} \quad \frac{dH}{dx_j}(t, x(t), y(t)) = -\dot{y}_j(t) \tag{8}$$

Thus, in summary, we have shown that if $x(t)$ is a solution to the Euler-Lagrange equation, and the Legendre condition is satisfied at $(t_0, x(t_0), \dot{x}(t_0))$, then there exists an interval around t_0 such that $x(t)$ and $y(t)$ satisfy the above differential equations, which we refer to as the *Hamilton equations*.

Conversely, suppose the Legendre condition of the Lagrangian L is satisfied at point (t_0, x_0, v_0). We define $G_j(t, x, y)$ as before, so that $y - \frac{dL}{dv}(t, x, G(t, x, y)) = 0$ on U about (t_0, x_0, y_0) with $y_0 = \frac{dL}{dv}(t_0, x_0, v_0)$. Suppose the functions $x(t)$ and $y(t)$ with $x(t_0) = x_0$ and $y(t_0) = y_0$ satisfy the Hamilton equations on some interval about t_0. Note that Eq. (5) and Eq. (8) imply $G(t, x(t), y(t)) = \dot{x}(t)$ so $y(t) = \frac{dL}{dv}(t, x(t), \dot{x}(t))$, and we have

$$0 = \dot{y}_j(t) - \frac{d}{dt}\frac{dL}{dv_j}(t, x(t), \dot{x}(t)) = -\frac{dH}{dx_j}(t, x(t), y(t)) - \frac{d}{dt}\frac{dL}{dv_j}(t, x(t), \dot{x}(t)) \tag{9}$$

$$= \frac{dL}{dx_j}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\frac{dL}{dv_j}(t, x(t), \dot{x}(t)) \tag{10}$$

where the final equality uses Eq. (6). This is precisely the Euler-Lagrange equation. Hence, we have proved the following claim:

Claim II.1. If L is a Lagrangian which satisfies the Legendre condition at some point (t_0, x_0, v_0), then trajectory $x(t)$ with $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$ satisfies the Euler-Lagrange equation on some open interval around t_0 if and only if there exists some $y(t)$ also defined on an interval around t_0 with $y(t_0) = y_0 = \frac{dL}{dv}(t_0, x_0, v_0)$ such that $x(t)$ and $y(t)$ satisfy the Hamilton equations, for the Hamiltonian H defined on some open set around (t_0, x_0, y_0). Moreover, when such a y exists, it is always the case that $y(t) = \frac{dL}{dv}(t, x(t), \dot{x}(t))$.

B. Solutions

Solution II.1 (Problem 1.1.5)**.** We are assuming that $(x, y) : I \to \mathbb{R}^{2n}$ is a trajectory such that

$$\dot{x}_j(t) = \frac{dH}{dy_j}(t, x(t), y(t)) \quad \text{and} \quad \dot{y}_j(t) = -\frac{dH}{dx_j}(t, x(t), y(t)) \tag{11}$$

We are also assuming that $\det \frac{d^2 H}{dy_i dy_j} \neq 0$. Consider the equation, $v = \frac{dH}{dy}(t, x, y)$, suppose (v_0, t_0, x_0, y_0) satisfies the equation. The condition and implicit function theorem implies that in a neighbourhood of (v_0, t_0, x_0), we can write $y = G(v, t, x)$ with $y_0 = G(v_0, t_0, x_0)$. We then define

$$L(t, x, v) = \sum_j v_j G_j(v, t, x) - H(t, x, G(v, t, x)) \tag{12}$$

in a neighbourhood of (t_0, x_0, v_0). Suppose we consider the point $(v_0, t_0, x_0, y_0) = (\dot{x}(t_0), t_0, x(t_0), y(t_0))$ for some t_0. Clearly, such a point will satisfy the above criterion, so we can define L in a neighbourhood of $(\dot{x}(t_0), t_0, x(t_0))$. We have

$$\frac{dL}{dv_j}(t, x(t), \dot{x}(t)) = G_j(\dot{x}(t), t, x(t)) + \sum_i \left[\dot{x}_i(t) \frac{dG_i}{dv_j}(\dot{x}(t), t, x(t)) - \frac{dH}{dy_i}(t, x(t), y(t)) \frac{dG_i}{dv_j}(\dot{x}(t), t, x(t)) \right] \tag{13}$$

$$= G_j(\dot{x}(t), t, x(t)) = y_j(t) \tag{14}$$

which means that

$$\frac{d}{dt}\frac{dL}{dv_j}(t, x(t), \dot{x}(t)) = \dot{y}_j(t) = -\frac{dH}{dx_j}(t, x(t), y(t)) \tag{15}$$

From here, note that

$$\frac{dL}{dx_j}(t, x(t), \dot{x}(t)) = \sum_i \left[\dot{x}_i(t) \frac{dG_i}{dx_j}(\dot{x}(t), t, x(t)) - \frac{dH}{dy_i}(t, x(t), y(t)) \frac{dG_i}{dx_j}(\dot{x}(t), t, x(t)) \right] - \frac{dH}{dx_j}(t, x(t), y(t)) \tag{16}$$

$$= -\frac{dH}{dx_j}(t, x(t), y(t)) \tag{17}$$

which implies that the prior equation is exactly the Euler-Lagrange equation, as desired.

Solution II.2 (Problem 1.1.20)**.** Note that

$$\omega_0(X_F, X_G) = dF(X_G) = X_G(F) = -(\nabla F)^T J_0 \nabla G = \{F, G\} \tag{18}$$

Thus, if $X = X_F$ and $Y = X_G$ are symplectic vector fields, then

$$X_{\omega_0(X,Y)} = X_{\{F,G\}} = [X_F, X_G] = [X, Y] \tag{19}$$

Solution II.3 (Problem 1.1.24)**.** In the particular case of geodesic flow, let us begin by noting that

$$\frac{d^2L(x,v)}{dv_i dv_j} = \frac{d^2}{dv_i dv_j}\frac{1}{2}\sum_{ij} g_{ij}(x)v_i v_j = \frac{1}{2}(g_{ij}(x) + g_{ji}(x)) = g_{ij}(x) \tag{20}$$

which means that the determinant is non-zero, as the metric is a non-degenerate symmetric two-tensor. Thus, we may apply the Legendre transform everywhere to get $y_k = \frac{dL}{dv_k} = \sum_j g_{kj}(x)v_j$. This means that $v_j = \sum_k g^{jk}(x)y_k$, and we get as our Hamiltonian

$$H(x,y) = \sum_j y_j v_j - L = \sum_j g^{jk}(x)y_j y_k - \frac{1}{2}\sum_{ijk\ell} g_{ij}(x)g^{ik}(x)g^{j\ell}(x)y_k y_\ell \tag{21}$$

$$= \sum_{jk} g^{jk}(x)y_j y_k - \frac{1}{2}\sum_{ik} g^{ik}(x)y_k y_i \tag{22}$$

$$= \frac{1}{2}\sum_{jk} g^{jk}(x)y_j y_k = \frac{1}{2}\langle y, g^{-1}(x)y\rangle \tag{23}$$

which implies that Hamilton's equations become

$$\dot{y}_k = -\frac{1}{2}\sum_{ij}\frac{dg^{ij}(x)}{dx_k}y_i y_j \quad \text{and} \quad \dot{x}_k = \frac{1}{2}\sum_j g^{jk}(x)y_j \tag{24}$$

To prove that $|\cdot|_g$ is constant along geodesics, we can show that the smooth function $\frac{1}{2}|v|_g^2 = \frac{1}{2}\langle v, g(x)v\rangle$ is constant along geodesics, $(x,v) = (x(t), \dot{x}(t))$. But of course, $v = g^{-1}(x)y$, so if $(x(t), y(t))$ is the phase space trajectory, then $\dot{x}(t) = g^{-1}(x(t))y(t)$, and

$$\frac{1}{2}|\dot{x}(t)|_g^2 = \langle g^{-1}(x(t))y(t), y(t)\rangle = H(x(t), y(t)) \tag{25}$$

We know that the Hamiltonian H will remain constant over geodesics, so the norm velocity will as well.

Solution II.4 (Problem 1.2.2)**.** Note that for Hamiltonian functions H and H' such that $S = H^{-1}(c) = (H')^{-1}(c')$ for regular values c and c', the Hamiltonian vector fields $X_H(z)$ and $X_{H'}(z)$ lie in the rank-one subbundle of T_zS given by $J_0 N_z S$, where $N_z S$ is the normal bundle at z. Note that $\omega(X_H(z), J_0 X_H(z)) = -||X_H(z)||^2$, so we can define $\lambda(z) = -\omega(X_{H'}(z), J_0 X_H(z))||X_H(z)||^{-2}$, provided that $X_H(z) \neq 0$, which we may assume as c is a regular value, implying H has non-zero derivative at every $z \in S$. We then have $X_{H'}(z) = \lambda(z)X_H(z)$. Note that λ is always non-zero. Consider the function $\lambda \circ z : \mathbb{R} \to \mathbb{R}$, where z is the solution of the H -system. By existence of uniqueness, the ODE $\dot{\tau}(t) = (\lambda \circ z)(\tau(t))$ has a unique solution with $\tau(0) = 0$ on the same time interval that z is defined on (all of $\mathbb{R}$). It is a reparametrization as $\dot{\tau}(t) \neq 0$, as $\lambda \circ z$ never is 0. To find T' such that $\tau(kT') = kT$ for all $k \in \mathbb{Z}$, note that we can of course find unique T' such that $\tau(T') = T$. Let $\sigma(t + kT') = \tau(t) + kT$ for $t \in [0, T']$ and $k \in \mathbb{Z}$. Note that $\sigma(kT') = \tau(0) + kT = kT$ while $\sigma(T' + (k-1)T') = \tau(T') + (k-1)T = kT$, so this function is well-defined and continuous. It is smooth on each interval $(kT', (k+1)T')$. In addition,

$$(\lambda \circ z)(\sigma(t + kT')) = \lambda(z(\tau(t) + kT)) = \lambda(z(\tau(t))) = \dot{\tau}(t) = \dot{\sigma}(t + kT') \tag{26}$$

on each of these open intervals. Hence, by uniqueness, $\sigma = \tau$ on each of the open intervals, and by continuity, they will be equal everywhere. In particular, we have $\sigma(kT') = kT$ for all k, as desired. It follows immediately that

$$\frac{d}{dt}(z \circ \tau)(t) = \dot{z}(\tau(t))\dot{\tau}(t) = \lambda(z(\tau(t)))X_H(z(\tau(t))) = X_{H'}(z(\tau(t))) \tag{27}$$

so that $z \circ \tau$ is a time-periodic solution of the H' -system, as desired.

III. Chapter 2

A. Notes

Definition III.1 (Fibration)**.** A fibration is a continuous map $p : E \to B$ satisfying the homotopy lifting property for all topological spaces X. This means that given any homotopy $F : X \times [0,1] \to B$ and any lift of $F(\cdot, 0)$ to E, there exists a homotopy lifting F which begins at the given lift of $F(\cdot, 0)$. A *Serre fibration* refers to the weaker condition where X need only be a CW-complex.

One of the main, useful properites of fibrations is that they fit into a long exact sequence of homotopy groups.

Claim III.1. The map $\det : U(n) \to S^1$ is a fibration.

Proof. □

1. Maslov index

We proved that the spaces $\mathrm{Sp}(2n)$ and $U(n)$ are homotopy equivalent, which implies that $\pi_1(\mathrm{Sp}(2n)) = \pi_1(U(n)) = \pi_1(S^1) \simeq \mathbb{Z}$. The Maslov index is an explicit homomorphism from $\pi_1(\mathrm{Sp}(2n), x_0)$, where x_0 is a fixed basepoint, to $\mathbb{Z}$. In particular, we have the map $f : \mathrm{Sp}(2n) \to U(n)$ which takes Ψ to

$$\Psi(\Psi^T\Psi)^{-1/2} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \simeq X + iY \in U(n) \tag{28}$$

We then take $X + iY$ to S^1 via $\det : U(n) \to S^1$. Note that f is a homotopy equivalence, so $f_* : \pi_1(\mathrm{Sp}(2n), x_0) \to \pi_1(U(n), f(x_0))$ is an isomorphism of fundamental groups. Similarly, it is proved in the textbook that $\det : U(n) \to S^1$ induces an isomorphism of fundamental groups via $\det_*$. Finally, we know that the degree of self-map of the circle is an isomorphism of $\pi_1(S^1, y_0)$ with $\mathbb{Z}$. Composing these maps together, $\deg \circ \det_* \circ f_*$, yields an isomorphism of $\pi_1(\mathrm{Sp}(2n), x_0)$ with $\mathbb{Z}$, as desired.

Claim III.2. The Maslov index is the unique map satisfying the provided axioms in the book.

Proof. Let μ be the Maslov index constructed above, let ψ be another map satisfying the same properties. Let $\Psi : S^1 \to \mathrm{Sp}(2n)$ be a loop based at the identity and let $m = \mu([\Psi]) \in \mathbb{Z}$. Define $U_m^{(n)} : S^1 \to U(n)$ as $U_m^{(n)}(t) = \mathrm{diag}(e^{imt}, 0, \dots, 0)$. It is easy to see that $\mu([U_m^{(n)}]) = m$ via the properties of μ, which means that $[U_m^{(n)}] = [\Psi]$ in the fundamental group. But this then implies via the properties we assumed that ψ satisfies,

$$\psi(\Psi) = \psi(U_m^{(n)}) = \psi(e^{imt}) + \psi(1) + \cdots + \psi(1) = m = \mu(\Psi) \tag{29}$$

so ψ and μ are equal. □

B. Solutions

Solution III.1 (2.1.2)**.** First suppose that $\Psi : V \to V$ is a linear symplectomorphism. Consider the vector space $V \times V$ and subspace Γ_Ψ. Note that

$$((-\omega) \oplus \omega)((v_1, v_2), (w_1, w_2)) = \omega(w_1, w_2) - \omega(v_1, v_2) = -((-\omega) \oplus \omega)((w_1, w_2), (v_1, v_2)) \tag{30}$$

so that $(-\omega) \oplus \omega$ is skew-symmetric. It is non-degenerate, as if $((-\omega) \oplus \omega)((v_1, \cdot), (w_1, \cdot)) = 0$ no matter what (v_2, w_2) we plug in, this is true for $(v_2, 0)$ and $(0, w_2)$, implying $v_1 = w_1 = 0$, as ω is non-degenerate. It follows that we have a valid symplectic form on our vector space. Note

$$((-\omega) \oplus \omega)((v, w), (\Psi v, \Psi w)) = \omega(\Psi v, \Psi w) - \omega(v, w) = 0 \tag{31}$$

so Γ_Ψ is isotropic. Since it is half the dimension of $V \times V$, it follows that it is Lagrangian. On the other hand, suppose Γ_Ψ is Lagrangian, so in particular, it is isotropic. Then from the above equation, $\Psi^*\omega(v,w) = \omega(v,w)$ for all $v, w \in V$. Thus, Ψ preserves the symplectic form. Moreover, if $\Psi(v) = 0$, then $\omega(v,w) = 0$ for all w, so $v = 0$, implying Ψ is injective, thus an isomorphism.

IV. Background material

A. Frobenius' theorem

In this section, we provide a concise proof of Frobenius' theorem, which is an important result relating to foliations (which arise frequently in symplectic topology).

B. Some homotopy theory

There are a few results in this book that require some homotopy-theoretic results, particularly related to fibrations and the associated homotopy long exact sequence.

C. Intersection number